THE BISHOP-PHELPS-BOLLOBÁS PROPERTY FOR NUMERICAL RADIUS IN $\ell_1(\mathbb{C})$

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ABSTRACT. We show that the set of bounded linear operators from X to X admits a Bishop-Phelps-Bollobás type theorem for numerical radius whenever X is $\ell_1(\mathbb{C})$ or $c_0(\mathbb{C})$. As an essential tool we provide two constructive versions of the classical Bishop-Phelps-Bollobás theorem for $\ell_1(\mathbb{C})$.

1. INTRODUCTION

The Bishop-Phelps theorem states that norm attaining functionals on a Banach space X are dense in its dual space X^*. In 1970, B. Bollobás extended this result in a quantitative way in order to work on problems related to the numerical range of an operator [Bol70]. One of the versions of his extension is presented below:

Theorem 1.1. *Let X be a Banach space. Given $\varepsilon > 0$, if $x \in X$, $x^* \in X^*$ with $\|x\| = \|x^*\| = 1$ and $x^*(x) \geq 1 - \frac{\varepsilon^2}{2}$, then there exist elements $x_0 \in X$ and $x_0^* \in X^*$ such that $\|x_0\| = \|x_0^*\| = x_0^*(x_0) = 1$,*

$$\|x - x_0\| \leq \varepsilon \text{ and } \|x^* - x_0^*\| \leq \varepsilon.$$

However, the known proofs of this fact have an existence nature –they are based on Hahn-Banach extension theorem, the Ekeland variational principle or Brøndsted-Rockafellar principle. In this paper we construct, as a necessary tool for our main results, explicit expressions of the approximating pair (x_0, x_0^*) when $X = \ell_1(\mathbb{C})$ –see Theorems 2.4 and 2.6.

Paralleling the research of norm attaining operators initiated by Lindenstrauss in [Lin63], B. Sims raised the question of the norm denseness of the set of numerical radius attaining operators –see [Sim72]. Partial positive results have been proved. We emphasize for their importance the results of M. Acosta in her Ph. D. thesis [Aco90], where a systematic study of the problem was initiated, the renorming result in [Aco93], and joint findings of this author with R. Payá [AP89, AP93]. Prior to them, I. Berg and B. Sims in [BS84] gave a positive answer for uniformly convex spaces and C. S. Cardassi obtained positive answers for ℓ_1, c_0, $C(K)$, $L_1(\mu)$, and uniformly smooth spaces [Car85a, Car85b, Car85c].

Using a renorming of c_0, R. Payá provided an example of a Banach space X such that the set of numerical radius attaining operators on X is not norm dense, answering in the negative Sims' question –see [Pay92]. In the same year, M. Acosta, F.

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Aguirre, and R. Payá in [AAP92] gave another counterexample: $X = \ell_2 \oplus_\infty G$, where G is the Gowers space.

Recently, M. Acosta *et al.* studied in [AAGM08] a new property, called the *Bishop-Phelps-Bollobás property for operators*, BPBp for short. A pair of Banach spaces (X, Y) has the BPBp if a "Bishop-Phelps-Bollobás" type theorem can be proved for the set of operators from X to Y. This property implies, in particular, that the norm attaining operators from X to Y are dense in the whole space of continuous linear operators $\mathfrak{L}(X, Y)$. However, as shown in [AAGM08], the converse is not true. Consequently, the BPB property is more than a quantitative tool for studying the density of norm attaining operators.

We investigate here an analogue of the Bishop-Phelps-Bollobás property for operators but in relation with numerical radius attaining operators. We call it the *Bishop-Phelps-Bollobás property for numerical radius*, BPBp-ν for short. The relation between norm attaining and numerical radius attaining operators is far from being clear, although the existence of an interconnection is evident. Accordingly, our goals in this paper are to define this new property –see Definition 1 below– and to show that $\ell_1(\mathbb{C})$ and $c_0(\mathbb{C})$ satisfy it –see Theorems 3.1 and 4.1. This brings an extension as well as a quantitative version of C. S. Cardassi's results in [Car85b].

Observe that the counterexamples provided in [AAP92] and [Pay92] imply, in particular, that there exist Banach spaces failing the Bishop-Phelps-Bollobás property for numerical radius.

Given a Banach space $(X, \|\cdot\|)$, we denote as usual by S_X and B_X, respectively, the unit sphere and the unit ball of X. By X^* we represent its dual, endowed with its standard norm $\|x^*\| = \sup_{x\in B_X}\{|x^*(x)|\}$ and by $\Pi(X)$ the set

$$\Pi(X) = \{(x, x^*) \in S_X \times S_{X^*} : \ x^*(x) = 1\}.$$

Given $x \in S_X$ and $x^* \in S_{X^*}$, we set

$$\pi_1(x^*) := \{x \in S_X : \ x^*(x) = 1\}.$$

By $\mathfrak{L}(X)$ we mean the Banach space of all linear and continuous operators from X into X endowed with its natural norm $\|T\| = \sup_{x\in B_X}\{\|Tx\|\}$. For a given $T \in \mathfrak{L}(X)$, its *numerical radius* $\nu(T)$ is defined by

$$\nu(T) = \sup\{|x^*(Tx)| : \ (x, x^*) \in \Pi(X)\}.$$

It is well known that the numerical radius of a Banach space X is a continuous seminorm on X which is, in fact, an equivalent norm when X is complex. In general, there exists a constant $n(X)$, called the *numerical index* of X, such that

$$n(X)\,\|T\| \le \nu(T) \le \|T\|\,, \text{ for all } T \in \mathfrak{L}(X).$$

Our interest in this paper is in spaces of numerical index 1, $n(X) = 1$, where the norm and the numerical radius coincide. For background in numerical radius we refer to the monographs [BD71, BD73] and in numerical index we refer to the survey [KMP06].

We say that $T \in \mathfrak{L}(X)$ attains its numerical radius if there exists $(x, x^*) \in \Pi(X)$ such that $|x^*(Tx)| = \nu(T)$. The set of numerical radius attaining operators will be denoted by $\mathrm{NRA(X)} \subset \mathfrak{L}\mathrm{(X)}$.

Definition 1 (BPBp-ν)**.** A Banach space X is said to have the *Bishop-Phelps-Bollobás property for numerical radius* if for every $0 < \varepsilon < 1$, there exists $\delta > 0$ such that for a given $T \in \mathfrak{L}(X)$ with $\nu(T) = 1$ and a pair $(x, x^*) \in \Pi(X)$

satisfying $|x^*(Tx)| \geq 1-\delta$, there exist $S \in \mathfrak{L}(X)$ with $\nu(S)=1$, and a pair $(y,y^*) \in \Pi(X)$ such that

$$\nu(T-S) \leq \varepsilon,\ \|x-y\| \leq \varepsilon,\ \|x^*-y^*\| \leq \varepsilon \ \text{ and } \ |y^*(Sy)|=1. \tag{1.1}$$

Observe that if X is a Banach space with $n(X)=1$, then the seminorm $\nu(\cdot)$ can be replaced by $\|\cdot\|$ in the definition above. Note that all the spaces studied in this paper have numerical index 1.

Notation and terminology. Throughout this paper $\arg(\cdot)$ stands for the function which sends a non zero complex number z to the unique $\arg(z) \in [0,2\pi)$ such that $z=|z|e^{\arg(z)i}$. For convenience we extend the function to $\mathbb{C}$ by writing $\arg(0)=0$. Following the standard notation, let $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ be, respectively, the real and imaginary part of the complex number $z \in \mathbb{C}$.

All along sections 2 to 4, the spaces ℓ_1, ℓ_∞, and c_0 stand respectively for $\ell_1(\mathbb{C})$, $\ell_\infty(\mathbb{C})$, and $c_0(\mathbb{C})$. The standard basis of ℓ_1 is denoted by $\{e_n\}_{n\in\mathbb{N}}$, and its biorthogonal functionals by $\{e_n^*\}_{n\in\mathbb{N}}$. Given a sequence $\xi=(\xi_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ and a complex function $f\colon \mathbb{C}\to\mathbb{C}$ we write $f(\xi)$ meaning the sequence $(f(\xi_j))_{j\in\mathbb{N}}$.

The following sets will be of help in the formulation of the results and proofs. Given $x=(x_j)_{j\in\mathbb{N}} \in \ell_1$, $\varphi=(\varphi_j)_{j\in\mathbb{N}} \in \ell_\infty$ we define

$$\mathcal{N}_{(x,\varphi)} = \{j \in \mathbb{N}\colon\ \varphi_j\, x_j = |x_j|\}, \tag{1.2}$$
$$\mathrm{supp}(x) = \{j \in \mathbb{N}\colon\ |x_j| \neq 0\}.$$

For $r>0$ we consider

$$\mathcal{A}_\varphi(r) = \{j \in \mathbb{N}\colon\ |\varphi_j| \geq 1-r\}, \tag{1.3}$$
$$\mathcal{P}_{(x,\varphi)}(r) = \{j \in \mathrm{supp}(x)\colon\ \mathrm{Re}(\varphi_j\, x_j) \geq (1-r)|x_j|\}. \tag{1.4}$$

Observe that $\mathcal{P}_{(x,\varphi)}(r) \subset \mathcal{A}_\varphi(r)$ and that if $x_j \geq 0$ for all $j\in\mathbb{N}$ –we describe this situation saying that x is *positive*– then

$$\mathcal{P}_{(x,\varphi)}(r) = \{j \in \mathrm{supp}(x)\colon\ \mathrm{Re}(\varphi_j) \geq (1-r)\}.$$

For a given set Γ, a subset $A \subset \Gamma$ and $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, we denote by $\mathbb{1}_A$ the characteristic function of A, that is, the element in $\mathbb{K}^\Gamma$ such that $(\mathbb{1}_A)_\gamma = 1$ if $\gamma \in A$ and $(\mathbb{1}_A)_\gamma = 0$ otherwise.

2. The Bishop-Phelps-Bollobás theorem in $\ell_1(\mathbb{C})$

In this section we present two constructive versions of Theorem 1.1, which are the main tool in the proof of our Theorems 3.1 and 5.1.

Lemma 2.1. *Let $(x,\varphi) \in S_{\ell_1} \times S_{\ell_\infty}$. Then $x \in \pi_1(\varphi)$ if and only if $\mathcal{N}_{(x,\varphi)} = \mathbb{N}$.*

Proof. Given a pair $(x,\varphi) \in S_{\ell_1} \times S_{\ell_\infty}$ satisfying $\mathcal{N}_{(x,\varphi)} = \mathbb{N}$, one can compute $\varphi(x) = \sum_{j\in\mathbb{N}} \varphi_j\, x_j \overset{(1.2)}{=} \sum_{j\in\mathbb{N}} |x_j| = \|x\| = 1$, which implies that $(x,\varphi) \in \Pi(\ell_1)$.

Conversely, let us assume that $(x,\varphi) \in \Pi(\ell_1)$ then,

$$1 = \mathrm{Re}(\varphi(x)) = \sum_{j\in\mathbb{N}} \mathrm{Re}(\varphi_j\, x_j) \leq \sum_{j\in\mathbb{N}} |\varphi_j\, x_j| \leq \sum_{j\in\mathbb{N}} |x_j| = 1,$$

which implies that $\mathrm{Re}(\varphi_j\, x_j) = |\varphi_j\, x_j| = |x_j|$ for $j\in\mathbb{N}$. Therefore, $\varphi_j\, x_j = |x_j|$ for every $j\in\mathbb{N}$, which finishes the proof. $\square$

Lemma 2.1 provides the essential insight into the properties of $\Pi(\ell_1)$ that we need for the proof of Theorems 2.4 and 2.6. A glance at Lemma 2.1 gives the following easy result regarding the norm attaining functionals on ℓ_1, $\mathrm{NA}(\ell_1)$.

Corollary 2.2. $\mathrm{NA}(\ell_1) = \{\varphi \in \ell_\infty \colon \exists \mathrm{n} \in \mathbb{N}$ *with* $|\varphi_{\mathrm{n}}| = \|\varphi\|\}$.

The following lemma is an adaptation of [AAGM08, Lemma 3.3] to our notation.

Lemma 2.3. *Let* $(x, \varphi) \in B_{\ell_1} \times B_{\ell_\infty}$ *and* $0 < \delta < 1$ *such that* $\varphi(x) \geq 1 - \delta$. *Then, for every* $\delta < r < 1$ *we have* $\left\|\mathrm{Re}\big(e^{\arg(\varphi) i}\, x\big) \cdot \mathbb{1}_{\mathcal{P}_{(x,\varphi)}(r)}\right\| \geq 1 - (\delta/r)$.

Proof. By assumption, we have that

$$\begin{aligned} 1 - \delta \leq \mathrm{Re}(\varphi(x)) &= \sum_{j \in \mathbb{N}} \mathrm{Re}(\varphi_j\, x_j) = \sum_{j\in\mathbb{N}} |\varphi_j|\, \mathrm{Re}\big(e^{\arg(\varphi_j) i}\, x_j\big) \\ &\leq \sum_{\mathcal{P}_{(x,\varphi)}(r)} \mathrm{Re}\big(e^{\arg(\varphi_j) i}\, x_j\big) + (1 - r) \sum_{\mathbb{N} \setminus \mathcal{P}_{(x,\varphi)}(r)} |x_j| \\ &\leq r \sum_{\mathcal{P}_{(x,\varphi)}(r)} \Big|\mathrm{Re}\big(e^{\arg(\varphi_j) i}\, x_j\big)\Big| + (1 - r), \end{aligned}$$

which implies that

$$\Big\|\mathrm{Re}\big(e^{\arg(\varphi) i}\, x\big)\, \mathbb{1}_{\mathcal{P}_{(x,\varphi)}(r)}\Big\| = \sum_{j \in \mathcal{P}_{(x,\varphi)}(r)} \Big|\mathrm{Re}\Big(e^{\arg(\varphi_j) i}\, x_j\Big)\Big| \geq 1 - (\delta/r),$$

as we wanted to show. □

Observe that the previous lemma implies, in particular, that

$$\left\|x \cdot \mathbb{1}_{\mathcal{P}_{(x,\varphi)}(r)}\right\| \geq 1 - (\delta/r).$$

We present next the two constructive versions of the Bishop-Phelps-Bollobás theorem.

2.1. **First constructive version.**

Theorem 2.4. *Given* $(x,\, \varphi) \in B_{\ell_1} \times B_{\ell_\infty}$ *and* $0 < \varepsilon < 1$ *such that* $\varphi(x) \geq 1 - \frac{\varepsilon^3}{4}$. *Then, there exists* $(x_0, \varphi_0) \in \Pi(\ell_1)$ *such that* $\|x - x_0\| \leq \varepsilon$, $\|\varphi - \varphi_0\| \leq \varepsilon$. *Moreover, we can take*

$$x_0 := \left\|x \cdot \mathbb{1}_{\mathcal{P}_{(x,\varphi)}(\varepsilon^2/2)}\right\|^{-1} \cdot x \cdot \mathbb{1}_{\mathcal{P}_{(x,\varphi)}(\varepsilon^2/2)}. \tag{2.1}$$

Proof. Set $P := \mathcal{P}_{(x,\varphi)}(\varepsilon^2/2)$ –see definition (1.4). Applying Lemma 2.3 with $\delta = \varepsilon^2/2$ and $r = \varepsilon$ gives that

$$M := \|x \cdot \mathbb{1}_P\| \geq 1 - (\varepsilon/2). \tag{2.2}$$

Let us define

$$\varphi_0 := \varphi \cdot \mathbb{1}_{\mathbb{N}\setminus P} + e^{-\arg(x) i} \cdot \mathbb{1}_P \in S_{\ell_\infty} \tag{2.3}$$

and

$$x_0 := M^{-1} x \cdot \mathbb{1}_P \in S_{\ell_1}. \tag{2.4}$$

On one hand, we can compute

$$\begin{aligned} \|x - x_0\| &\overset{(2.4)}{=} \left\|x - M^{-1} x \cdot \mathbb{1}_P\right\| = (M^{-1} - 1)\, \|x \cdot \mathbb{1}_P\| + \left\|x \cdot \mathbb{1}_{\mathbb{N}\setminus P}\right\| \\ &\overset{(2.2)}{=} (1 - M) + \left\|x \cdot \mathbb{1}_{\mathbb{N}\setminus P}\right\| \overset{\|x\|\leq 1}{\leq} 2 - 2M \overset{(2.2)}{\leq} \varepsilon, \end{aligned}$$

and, since the support of x_0 is included in P –this is a consequence of (2.4), we deduce that

$$\varphi_0(x_0) = \sum_{j\in P} (\varphi_0)_j\,(x_0)_j \overset{(2.3)}{=} \sum_{j\in P} e^{-\arg(x_j)i}\,(x_0)_j \overset{(2.4)}{=} \sum_{j\in P} |(x_0)_j| = \|x_0\| = 1,$$

which is equivalently expressed as $(x_0, \varphi_0) \in \Pi(\ell_1)$.

On the other hand, using that

$$|z-1| \leq \sqrt{2(1-\mathrm{Re}(z))} \text{ for every } z \in \mathbb{C} \text{ such that } |z| \leq 1, \tag{2.5}$$

we deduce

$$\begin{aligned}\|\varphi - \varphi_0\| &\overset{(2.3)}{=} \sup_{j\in P}\{|\varphi_j - (\varphi_0)_j|\} \overset{(2.3)}{=} \sup_{j\in P}\left\{\left|\varphi_j - e^{-\arg(x_j)i}\right|\right\}\\ &= \sup_{j\in P}\left\{\left|e^{\arg(x_j)i}\,\varphi_j - 1\right|\right\} \overset{(2.5)}{\leq} \sup_{j\in P}\left\{\sqrt{2-2\,\mathrm{Re}\big(e^{\arg(x_j)i}\,\varphi_j\big)}\right\}\\ &\leq \sqrt{2-2(1-\varepsilon^2/2)} = \varepsilon,\end{aligned}$$

which finishes the proof. □

An immediate consequence of Theorem 2.4 is the following version of the Bishop-Phelps-Bollobás theorem for $\ell_1(\mathbb{C})$.

Corollary 2.5. *Let $0 < \varepsilon < 1$ and $(x, \varphi) \in B_{\ell_1} \times B_{\ell_\infty}$ such that $|\varphi(x)| \geq 1 - \frac{\varepsilon^3}{4}$. Then, there exists $(x_0, \varphi_0) \in S_{\ell_1} \times S_{\ell_\infty}$ such that $\|x - x_0\| \leq \varepsilon$, $\|\varphi - \varphi_0\| \leq \varepsilon$ and $|\varphi_0(x_0)| = 1$.*

Proof. Apply Theorem 2.4 to the pair $\big(e^{-\arg(\varphi(x))i}\,x, \varphi\big)$ obtaining (z_0, φ_0) belonging to $\Pi(\ell_1)$ such that $\big\|e^{-\arg(\varphi(x))i}\,x - z_0\big\| \leq \varepsilon$ and $\|\varphi - \varphi_0\| \leq \varepsilon$. Therefore, if we set $x_0{:=}e^{\arg(\varphi(x))i}\,z_0$, the pair (x_0, φ_0) satisfies the conclusions of the corollary. □

2.2. **Second constructive version.** Given a pair (x, φ) and $0 < \varepsilon < 1$, Theorem 2.4 ensures the existence of a pair (x_0, φ_0) –defined by (2.4) and (2.3)– satisfying the conclusions of the Bishop-Phelps-Bollobás theorem. However, φ_0 depends on x, in fact, on $\arg(x)$. In order to prove Theorem 3.1 we will need a functional φ_0 depending only on the given ε and φ. So, we present the following result.

Theorem 2.6. *Let $(x, \varphi) \in B_{\ell_1} \times B_{\ell_\infty}$ and $0 < \varepsilon < 1$ be such that $\varphi(x) \geq 1 - \frac{\varepsilon^3}{60}$. Then there exists $(x_0, \varphi_0) \in \Pi(\ell_1)$ such that $\|x - x_0\| \leq \varepsilon$, $\|\varphi - \varphi_0\| \leq \varepsilon$. Moreover, the functional φ_0 can be defined as*

$$\varphi_0 = \varphi \cdot \mathbb{1}_{\mathbb{N}\setminus\mathcal{A}_\varphi(\varepsilon^2/20)} + e^{\arg(\varphi)i} \cdot \mathbb{1}_{\mathcal{A}_\varphi(\varepsilon^2/20)}. \tag{2.6}$$

Proof. Let us consider the isometry $S\colon \ell_1 \to \ell_1$ defined by

$$\langle e_j^*, Sy\rangle = e^{\arg(\varphi_j)i}\,y_j, \text{ for } y \in \ell_1 \text{ and } j \in \mathbb{N}. \tag{2.7}$$

Set $\widetilde{x} = Sx$ and $\widetilde{\varphi} = \varphi \circ S^{-1}$. Then, it is clear that the pair $(\widetilde{x}, \widetilde{\varphi})$ is in $B_{\ell_1} \times B_{\ell_\infty}$, that $\widetilde{\varphi}(\widetilde{x}) \geq 1 - \frac{\varepsilon^3}{60}$ and that $\widetilde{\varphi} = (|\varphi_j|)_{j\in\mathbb{N}}$ is positive. Denote by A and P respectively the sets $\mathcal{A}_{\widetilde{\varphi}}(r)$ and $\mathcal{P}_{(\widetilde{x},\widetilde{\varphi})}(r)$ –see definitions (1.3) and (1.4), where $r{:=}\frac{\varepsilon^2}{20}$. Let us define

$$\widehat{\varphi}{:=}\widetilde{\varphi} \cdot \mathbb{1}_{\mathbb{N}\setminus A} + \mathbb{1}_A \in S_{\ell_\infty} \tag{2.8}$$

and

$$\widehat{x}:=M^{-1}\mathrm{Re}(\widetilde{x})\cdot\mathbb{1}_P\in S_{\ell_1}, \tag{2.9}$$

where $M:=\|\mathrm{Re}(\widetilde{x})\cdot\mathbb{1}_P\|$. Applying Lemma 2.3 with $\delta=\varepsilon^3/60$ and r, gives that $M\geq 1-\frac{\varepsilon}{3}$. In particular, this means that P, and thus A, are non-empty.

We can compute that

$$\begin{aligned}\|\widetilde{\varphi}-\widehat{\varphi}\| &\overset{(2.8)}{=} \sup_{j\in A}\{|\widetilde{\varphi}_j-\widehat{\varphi}_j|\} \overset{(2.8)}{=} \sup_{j\in A}\{|\widetilde{\varphi}_j-1|\}\\ &= \sup_{j\in A}\{(1-\widetilde{\varphi}_j)\} \overset{(1.3)}{\leq} r\leq\varepsilon, \end{aligned}\tag{2.10}$$

and, since by (1.4) and (2.9) the support of $\widehat{x}$ is $P\subset A$ –which, in particular, implies that $\widehat{x}_j>0$ for $j\in P$, we deduce that

$$\widehat{\varphi}(\widehat{x})=\sum_{j\in P}\widehat{\varphi}_j\widehat{x}_j \overset{(2.8)}{=} \sum_{j\in P}\widehat{x}_j=\sum_{j\in P}|\widehat{x}_j|=1, \tag{2.11}$$

which is equivalently written as $(\widehat{x},\widehat{\varphi})\in\Pi(\ell_1)$.

In order to show that $\|\widetilde{x}-\widehat{x}\|\leq\varepsilon$, let us observe first that

$$\|\widetilde{x}\cdot\mathbb{1}_P\|=\sum_{j\in P}|\widetilde{x}_j|\geq\sum_{j\in P}|\mathrm{Re}(\widetilde{x}_j)|=M\geq 1-\frac{\varepsilon}{3}, \tag{2.12}$$

from which

$$\begin{aligned}\|\widetilde{x}-\widehat{x}\| &\overset{(2.9)}{=} \left\|\widetilde{x}-M^{-1}\mathrm{Re}(\widetilde{x})\cdot\mathbb{1}_P\right\| = \left\|\widetilde{x}\cdot\mathbb{1}_{\mathbb{N}\setminus P}\right\|+\left\|(\widetilde{x}-M^{-1}\mathrm{Re}(\widetilde{x}))\cdot\mathbb{1}_P\right\|\\ &\overset{(2.12)}{\leq} \frac{\varepsilon}{3}+\left\|(\widetilde{x}-M^{-1}\mathrm{Re}(\widetilde{x}))\cdot\mathbb{1}_P\right\|. \end{aligned}\tag{2.13}$$

We need a bit more care to estimate the last term in (2.13). From the very definition of P, we know that for every $j\in P$ it holds

$$|\widetilde{x}_j|\leq(1-r)^{-1}\widetilde{\varphi}_j\,\mathrm{Re}(\widetilde{x}_j). \tag{2.14}$$

Therefore,

$$\begin{aligned}\|(\widetilde{x}-\mathrm{Re}(\widetilde{x}))\cdot\mathbb{1}_P\| &= \sum_{j\in P}|\widetilde{x}_j-\mathrm{Re}(\widetilde{x}_j)|=\sum_{j\in P}|\mathrm{Im}(\widetilde{x}_j)|\\ &=\sum_{j\in P}\sqrt{|\widetilde{x}_j|^2-\mathrm{Re}(\widetilde{x}_j)^2}\\ &\overset{(2.14)}{\leq}\sum_{j\in P}|\mathrm{Re}(\widetilde{x}_j)|\sqrt{(1-r)^{-2}-1}\\ &\leq\|\widetilde{x}\|\sqrt{(1-r)^{-2}-1}\overset{r=\frac{\varepsilon^2}{20}}{\leq}\frac{\varepsilon}{3}, \end{aligned}\tag{2.15}$$

which implies that

$$\begin{aligned}\left\|(\widetilde{x}-M^{-1}\mathrm{Re}(\widetilde{x}))\cdot\mathbb{1}_P\right\| &\leq \|(\widetilde{x}-\mathrm{Re}(\widetilde{x}))\cdot\mathbb{1}_P\|+\left\|(1-M^{-1})\mathrm{Re}(\widetilde{x})\cdot\mathbb{1}_P\right\|\\ &\overset{(2.15)}{\leq}\frac{\varepsilon}{3}+(M^{-1}-1)\,\|\mathrm{Re}(\widetilde{x})\cdot\mathbb{1}_P\|\\ &=\frac{\varepsilon}{3}+(1-M)\leq\frac{2\varepsilon}{3}. \end{aligned}\tag{2.16}$$

Putting together (2.13) and (2.16), one obtains

$$\|\widetilde{x}-\widehat{x}\| \leq \frac{\varepsilon}{3} + \left\|(\widetilde{x} - M^{-1}\mathrm{Re}(\widetilde{x}))\cdot \mathbb{1}_P\right\| \leq \varepsilon, \tag{2.17}$$

which finishes the core of the proof.

Now, we define

$$x_0{:=}S^{-1}\widehat{x} \quad \text{and} \quad \varphi_0 = S^*(\widehat{\varphi}) = \widehat{\varphi}\circ S, \tag{2.18}$$

which by (2.11) gives that $\varphi_0(x_0) = \widehat{\varphi}(\widehat{x}) = 1$. Since S and S^* are isometries, we deduce from (2.10), (2.17), (2.18) and the definition of $\widetilde{x}$ and $\widetilde{\varphi}$ that

$$\|x - x_0\| \leq \varepsilon,\ \ \|\varphi - \varphi_0\| \leq \varepsilon.$$

Therefore, (x_0, φ_0) is the pair in $\Pi(\ell_1)$ we were looking for.

Bearing in mind (2.18), one computes

$$(\varphi_0)_j = \varphi_0(e_j) \overset{(2.18)}{=} \widehat{\varphi}(Se_j) \overset{(2.7)}{=} \widehat{\varphi}\big(e^{\arg(\varphi_j)i}\, e_j\big) = e^{\arg(\varphi_j)i}\,\widehat{\varphi}_j,$$

which together with (2.8) implies that $\varphi_0 = \varphi\cdot\mathbb{1}_{\mathbb{N}\setminus A} + e^{\arg(\varphi)i}\cdot\mathbb{1}_A$. Finally, noting that $A = \mathcal{A}_{\widetilde{\varphi}}(r) = \mathcal{A}_{\varphi}(r)$, the validity of (2.6) has been shown. □

Remark 2.7. Observe that the function φ_0 provided by Theorem 2.6 and defined by (2.6) only depends on ε and φ itself as well as satisfies $\pi_1(\varphi) \subset \pi_1(\varphi_0)$.

3. BPB PROPERTY FOR NUMERICAL RADIUS IN $\ell_1(\mathbb{C})$

As a consequence of Theorems 2.4 and 2.6 we show that ℓ_1 has the Bishop-Phelps-Bollobás property for numerical radius.

Theorem 3.1. *Let $T \in S_{\mathfrak{L}(\ell_1)}$, $0 < \varepsilon < 1$ and $(x, \varphi) \in \Pi(\ell_1)$ such that $\varphi(Tx) \geq 1 - (\varepsilon/9)^{9/2}$. Then there exist $T_0 \in S_{\mathfrak{L}(\ell_1)}$ and $(x_0, \varphi_0) \in \Pi(\ell_1)$ such that*

$$\|T - T_0\| \leq \varepsilon,\ \|x - x_0\| \leq \varepsilon,\ \|\varphi - \varphi_0\| \leq \varepsilon \textit{ and } \varphi_0(T_0x_0) = 1. \tag{3.1}$$

Proof. First of all, fix $\mu := \sqrt{\varepsilon^3/240}$. Using a suitable isometry, we can assume that x is positive. In particular, by Lemma 2.1 and the definition of $\mathcal{N}_{x,\varphi}$ in (1.2), we can assume that $\varphi_j = 1$ for $j \in \mathrm{supp}(x)$. Since $\mu^3/4 \geq (\varepsilon/9)^{9/2}$, Theorem 2.4 can be applied to the pair $(x,\, T^*\varphi) \in B_{\ell_1} \times B_{\ell_\infty}$ and μ instead of ε giving $x_0 \in \pi_1(\varphi)$ such that $\|x - x_0\| \leq \mu \leq \varepsilon$. Moreover, by (2.1) we know that

$$x_0 = \|x\cdot\mathbb{1}_P\|^{-1}\cdot x\cdot\mathbb{1}_P, \tag{3.2}$$

where the non-empty set P is defined by

$$P := \mathcal{P}_{(x,T^*\varphi)}(\mu^2/2) = \{j \in \mathrm{supp}(x)\colon \mathrm{Re}(T^*\varphi(e_j)) \geq 1 - \mu^2/2\}. \tag{3.3}$$

In particular, x_0 is positive.

Since $\mu^2/2 = \frac{(\varepsilon/2)^3}{60}$, for each $j \in P$ we can apply Theorem 2.6 to the pair $(e^{-\arg(\varphi(Te_j))i}\,Te_j, \varphi)$ and $\varepsilon/2$ to find $(z_j, \varphi_0) \in \Pi(\ell_1)$ such that

$$\|Te_j - a_jz_j\| \leq \varepsilon/2, \quad \|\varphi - \varphi_0\| \leq \varepsilon/2$$

and $\Pi_1(\varphi) \subset \Pi_1(\varphi_0)$ –see Remark 2.7, where $a_j = e^{\arg(\varphi(Te_j))i}$. Observe that φ_0 can be chosen independently on $j \in P$ and by (2.6) explicitly written as

$$\varphi_0 = \varphi\cdot\mathbb{1}_{\mathbb{N}\setminus\mathcal{A}_\varphi(\varepsilon^2/80)} + e^{\arg(\varphi)i}\cdot\mathbb{1}_{\mathcal{A}_\varphi(\varepsilon^2/80)}. \tag{3.4}$$

Let us define T_0 as the unique operator in $\mathfrak{L}(\ell_1)$ such that $T_0 e_i = Te_i$ for $i \notin P$ and $T_0 e_j = z_j$ for $j \in P$. Equivalently,

$$T_0 x = \mathbb{1}_{\mathbb{N}\setminus P} \cdot Tx + \sum_{j\in P} e_j^*(x) z_j, \text{ for } x \in \ell_1. \tag{3.5}$$

It is clear from (3.5) that

$$\|T_0\| = \sup_{n\in\mathbb{N}}\{\|T_0 e_n\|\} = \max\left\{\sup_{j\notin P}\{\|Te_j\|\}, \sup_{j\in P}\{\|z_j\|\}\right\} = 1.$$

Given $j \in P$, the identity (3.3) ensures that $\mathrm{Re}(\varphi(Te_j)) \geq 1 - \mu^2/2$. Using again the general fact (2.5), we deduce that $|a_j - 1| \leq \mu \leq \varepsilon/2$.

Therefore,

$$\begin{aligned}\|T - T_0\| &= \sup_{n\in\mathbb{N}}\{\|Te_n - T_0 e_n\|\} = \sup_{j\in P}\{\|Te_j - z_j\|\} \\ &\leq \sup_{j\in P}\{\|Te_j - a_j z_j\|\} + \sup_{j\in P}\{\|a_j z_j - z_j\|\} \\ &\leq \frac{\varepsilon}{2} + \sup_{j\in P}\{|a_j - 1|\} \leq \varepsilon.\end{aligned}$$

Since $x_0 \in \pi_1(\varphi)$ and $\pi_1(\varphi) \subset \pi_1(\varphi_0)$, we deduce that (x_0, φ_0) belongs to $\Pi(\ell_1)$. It remains to show that $\varphi_0(T_0 x_0) = 1$ to prove the validity of (3.1). But, since x_0 is positive, we obtain that

$$\begin{aligned}\varphi_0(T_0 x_0) &\overset{(3.5)}{=} \sum_{j\in P}(x_0)_j \varphi_0(z_j) + \sum_{j\notin P}(x_0)_j \varphi_0(Te_j) \\ &\overset{(3.2)}{=} \sum_{j\in P}(x_0)_j = \sum_{j\in P}|(x_0)_j| = \|x_0\| = 1,\end{aligned}$$

and the proof is over. □

Remark 3.2. We cannot replace the condition $(x, \varphi) \in \Pi(\ell_1)$ in Theorem 3.1 by the more general $(x, \varphi) \in B_{\ell_1} \times B_{\ell_\infty}$. Indeed, let us consider the operator $T\colon \ell_1 \to \ell_1$ defined by $Te_j = e_j$ for $j \geq 2$ and $Te_1 = e_2$. Take $(e_1, e_2^*) \in B_{\ell_1} \times B_{\ell_\infty}$, $T_0 \in \mathfrak{L}(\ell_1)$, and $(x, \varphi) \in B_{\ell_1} \times B_{\ell_\infty}$ such that $\|T - T_0\| \leq \varepsilon$, $\|e_1 - x\| \leq \varepsilon$, and $\|e_2^* - \varphi\| \leq \varepsilon$. Then

$$|\varphi(x)| \leq |\varphi(x) - e_2^*(x)| + |e_2^*(x) - e_2^*(e_1)| + |e_2^*(e_1)| \leq 2\varepsilon,$$

which implies that (x, φ) cannot be in $\Pi(\ell_1)$.

Corollary 3.3. *The Banach space ℓ_1 has the Bishop-Phelps-Bollobás property for numerical radius.*

Proof. Let us consider $T \in \mathfrak{L}(\ell_1)$ with $\nu(T) = 1$ and $0 < \varepsilon < 1$. Let us take a pair $(x, \varphi) \in \Pi(\ell_1)$ such that $|\varphi(Tx)| \geq 1 - (\varepsilon/9)^{\frac{9}{2}}$. In fact, we can assume that $\varphi(Tx) \geq 1 - (\varepsilon/9)^{\frac{9}{2}}$; otherwise, we proceed with $\widetilde{T} = e^{-\arg(\varphi(Tx))i}\,T$. Then Theorem 3.1 gives the existence of an operator $T_0 \in S_{\mathfrak{L}(\ell_1)}$ and a pair $(x_0, \varphi_0) \in \Pi(\ell_1)$ that satisfy conditions in (3.1), which are precisely the requirements (1.1) in Definition 1. □

Corollary 3.4 ([Car85b])**.** *The set* $\mathrm{NRA}(\ell_1)$ *is dense in* $\mathfrak{L}(\ell_1)$.

4. BPB PROPERTY FOR NUMERICAL RADIUS IN $c_0(\mathbb{C})$

Theorem 3.1 allows us to show that c_0 has the Bishop-Phelps-Bollobás property for numerical radius as well. Indeed, we rely on the fact that our constructions in ℓ_1 can be dualized.

Theorem 4.1. *Let $T \in S_{\mathfrak{L}(c_0)}$, $0 < \varepsilon < 1$ and $(x, \varphi) \in \Pi(c_0)$ such that $|\varphi(Tx)| \geq 1 - (\varepsilon/9)^{9/2}$. Then there exist $S \in S_{\mathfrak{L}(c_0)}$ and $(x_0, \varphi_0) \in \Pi(c_0)$, such that*

$$\|T - S\| \leq \varepsilon,\ \|x - x_0\| \leq \varepsilon,\ \|\varphi - \varphi_0\| \leq \varepsilon \text{ and } \varphi_0(Sx_0) = 1.$$

Proof. Throughout this proof we identify the elements in c_0 with their image in ℓ_∞ through the natural embedding $c_0 \to \ell_\infty$. The adjoint operator of T, $T^*\colon \ell_1 \to \ell_1$ satisfies

$$|x(T^*\varphi)| = |T^*(\varphi)(x)| = |\varphi(Tx)| \geq 1 - (\varepsilon/9)^{9/2}.$$

Without loss of generality, we can assume that $x(T^*\varphi) \geq 1 - (\varepsilon/9)^{9/2}$. Otherwise, employing techniques from the proof of Corollary 3.3, define the operator $\widetilde{T} = e^{-\arg(x(T^*\varphi))i}\, T^*$ and proceed with the proof for $x(\widetilde{T}\varphi) = |x(T^*\varphi)|$.

By Theorem 3.1, there exists $T_0 \in \mathfrak{L}(\ell_1)$, $\|T_0\| = 1$ and $(\varphi_0, x_0) \in \Pi(\ell_1)$ such that

$$\|T^* - T_0\| \leq \varepsilon, \quad \|\varphi - \varphi_0\| \leq \varepsilon, \quad \|x - x_0\| \leq \varepsilon$$

and $x_0(T_0\varphi_0) = 1$.

We assert that (x_0, φ_0) is the pair we are looking for. To show this, we will reexamine the proof of Theorem 3.1 to establish how x_0, φ_0 and T_0 are defined. Indeed, from (3.3), (3.2), (3.4) and (3.5) we have respectively

$$\begin{aligned} P &= \mathcal{P}_{(\varphi, T^{**}x)}(\varepsilon^3/480), \\ \varphi_0 &= \|\varphi \cdot \mathbb{1}_P\|^{-1} \cdot \varphi \cdot \mathbb{1}_P, \\ x_0 &= x \cdot \mathbb{1}_{\mathbb{N} \setminus A_x(\varepsilon^2/80)} + e^{\arg(x)i} \cdot \mathbb{1}_{A_x(\varepsilon^2/80)}, \\ T_0 x &= \mathbb{1}_{\mathbb{N} \setminus P} \cdot Tx + \sum_{j \in P} e_j^*(x) z_j, \text{ for } x \in \ell_1, \end{aligned} \tag{4.1}$$

where $\{z_j\}_{j \in P} \subset \pi_1(\varphi_0)$.

Note that $\mathcal{A}_x(\varepsilon^2/80) = \{j \in N\colon |x_j| \geq 1 - \varepsilon^2/80\}$ and that $x \in c_0$. Thus, $A_x(\varepsilon^2/80)$ is finite which, by (4.1), implies that $x_0 \in c_0$.

We shall show that T_0 is an adjoint operator and thus that there exists $S \in \mathfrak{L}(c_0)$ such that $S^* = T_0$. It will be enough to show that $T_0^*|_{c_0} \subset c_0$. Set $t_{ij} = \langle e_i, T(e_j) \rangle$ for $i, j \in \mathbb{N}$. Fix $i \in \mathbb{N}$, then for $j \in \mathbb{N}$

$$\langle e_j, T_0^*(e_i) \rangle = \begin{cases} t_{ji} & \text{if } j \notin P, \\ (z_j)_i & \text{if } j \in P. \end{cases}$$

Since $x \in c_0$, $T^{**}x$ belongs to c_0, which implies that P is finite. Accordingly, only finitely many terms of the form $\langle e_j, T_0^*(e_i) \rangle$ differ from the corresponding t_{ji}. On the other hand, since T belongs to $\mathfrak{L}(c_0)$, it holds that $\lim_j |t_{ji}| = 0$. Therefore, we deduce that $|\langle e_j, T_0^*(e_i) \rangle| \to 0$ when $j \to \infty$. This implies that $T_0^* e_i \in c_0$ and, since $i \in \mathbb{N}$ is arbitrarily chosen, we deduce that $T_0^*|_{c_0} \subset c_0$.

Hence we obtain the operator $S = T_0^*|_{c_0} \in \mathfrak{L}(c_0)$ and the pair $(x_0, \varphi_0) \in \Pi(c_0)$ satisfying:

$$\varphi_0(Sx_0) = S^*\varphi_0(x_0) = x_0(S^*\varphi_0) = x_0(T_0\varphi_0) = 1,$$

and

$$\|S - T\| = \|(S - T)^*\| = \|S^* - T^*\| = \|T_0 - T^*\| \leq \varepsilon,$$

which finishes the proof. □

Theorem 4.1 implies the following two corollaries.

Corollary 4.2. *The Banach space c_0 has the Bishop-Phelps-Bollobás property for numerical radius.*

Corollary 4.3 ([Car85b]). *The set $\mathrm{NRA}(c_0)$ is dense in $\mathfrak{L}(c_0)$.*

5. Generalizations and remarks

All the results that have been presented in sections 2, 3 and 4 were stated and proved for the Banach spaces $\ell_1(\mathbb{C})$ or $c_0(\mathbb{C})$. However, a glance at their proofs suffices to convince oneself of their validity for $\ell_1(\mathbb{R})$ and $c_0(\mathbb{R})$ –shorter proofs and better estimates can be obtained in this case. More generally, given a non-empty set Γ and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, these results are, after suitable adjustments, still valid for $\ell_1(\Gamma, \mathbb{K})$ and $c_0(\Gamma, \mathbb{K})$. The spaces $\ell_1(\Gamma, \mathbb{K})$ and $c_0(\Gamma, \mathbb{K})$ are, respectively, the ℓ_1-sum and the c_0-sum of Γ copies of the field $\mathbb{K}$. Note that in particular $\ell_1(\mathbb{N}, \mathbb{K}) = \ell_1(\mathbb{K})$.

The Banach space $c_0(\Gamma, \mathbb{K})$ is a predual of $\ell_1(\Gamma, \mathbb{K})$. Observe that both $c_0(\Gamma, \mathbb{K})$ and $\ell_1(\Gamma, \mathbb{K})$ have numerical index 1. Previous considerations imply that both of them also have the BPB property for numerical radius. The ω^* topology of $\ell_1(\Gamma, \mathbb{K})$ stands here for the topology induced on $\ell_1(\Gamma, \mathbb{K})$ by pointwise convergence on elements of $c_0(\Gamma, \mathbb{K})$.

On the other hand, the proof of Theorem 4.1 shows that in Theorem 3.1 we proved more than was stated. Indeed, putting together Theorem 3.1, the ideas on duality in the proof of Theorem 4.1 and considerations above, one easily proves the following theorem.

Theorem 5.1. *Let $T \in S_{\mathfrak{L}(\ell_1(\Gamma,\mathbb{K}))}$, $0 < \varepsilon < 1$ and $(x, \varphi) \in \Pi(\ell_1(\Gamma, \mathbb{K}))$ such that $\varphi(Tx) \geq 1 - (\varepsilon/9)^{9/2}$. Then there exist $T_0 \in S_{\mathfrak{L}(\ell_1(\Gamma,\mathbb{K}))}$ and $(x_0, \varphi_0) \in \Pi(\ell_1(\Gamma, \mathbb{K}))$ such that*

$$\|T - T_0\| \leq \varepsilon, \ \|x - x_0\| \leq \varepsilon, \ \|\varphi - \varphi_0\| \leq \varepsilon \ \text{ and } \ \varphi_0(T_0 x_0) = 1.$$

Moreover, if T is ω^*-ω^*-continuous and φ is ω^*-continuous, then T_0 and φ_0 will be ω^*-ω^*-continuous and ω^*-continuous, respectively.

Below are two consequences of Theorem 5.1.

Theorem 5.2. *The Banach space $\ell_1(\Gamma, \mathbb{K})$ has the BPB property for numerical radius.*

Theorem 5.3. *The Banach space $c_0(\Gamma, \mathbb{K})$ has the BPB property for numerical radius.*

Proof. Fix $0 < \varepsilon < 1$, $\delta \leq (\varepsilon/9)^{9/2}$, $T \in S_{\mathfrak{L}(c_0(\Gamma,\mathbb{K}))}$ and $(x, x^*) \in \Pi(c_0(\Gamma, \mathbb{K}))$ such that $x^*(Tx) \geq 1 - \delta$. Applying Theorem 5.1 to the ω^*-ω^*-continuous operator $T^* \in S_{\mathfrak{L}(\ell_1(\Gamma,\mathbb{K}))}$, the pair (x^*, x) and ε, gives a new $T_0 \in S_{\mathfrak{L}(c_0(\Gamma,\mathbb{K}))}$ and a new pair $(x_0^*, x_0^{**}) \in \Pi(\ell_1(\Gamma, \mathbb{K}))$ satisfying

$$\left\|T^* - T_0^*\right\| \leq \varepsilon, \ \|x - x_0^{**}\| \leq \varepsilon, \ \|x^* - x_0^*\| \leq \varepsilon \text{ and } x_0^{**}\left(T_0^* x_0^*\right) = 1. \tag{5.1}$$

Moreover, x_0^{**} is ω^*-continuous, so we can identify it with some $x_0 \in S_{c_0(\Gamma,\mathbb{K})}$. Therefore, conditions in (5.1) become

$$\big\|T - T_0\big\| \leq \varepsilon, \ \|x - x_0\| \leq \varepsilon, \ \|x^* - x_0^*\| \leq \varepsilon \text{ and } x_0^*\big(T_0 x_0\big) = 1.$$

which are the requirements (1.1) in Definition 1. Consequently, $c_0(\Gamma,\mathbb{K})$ has the Bishop-Phelps-Bollobás property for numerical radius. □

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References

[AAGM08] María D. Acosta, Richard M. Aron, Domingo García, and Manuel Maestre, *The Bishop-Phelps-Bollobás theorem for operators*, J. Funct. Anal. **254** (2008), no. 11, 2780–2799. MR 2414220 (2009c:46016)

[AAP92] María D. Acosta, Francisco J. Aguirre, and Rafael Payá, *A space by W. Gowers and new results on norm and numerical radius attaining operators*, Acta Univ. Carolin. Math. Phys. **33** (1992), no. 2, 5–14. MR 1287219 (95i:47009)

[Aco90] M. Acosta, *Operadores que alcanzan su radio numérico*, Ph.D. thesis, Universidad de Granada, 1990.

[Aco93] María D. Acosta, *Every real Banach space can be renormed to satisfy the denseness of numerical radius attaining operators*, Israel J. Math. **81** (1993), no. 3, 273–280. MR 1231192 (94h:47003)

[AP89] María D. Acosta and Rafael Paya, *Denseness of operators whose second adjoints attain their numerical radii*, Proc. Amer. Math. Soc. **105** (1989), no. 1, 97–101. MR 937841 (89f:47008)

[AP93] María D. Acosta and Rafael Payá, *Numerical radius attaining operators and the Radon-Nikodým property*, Bull. London Math. Soc. **25** (1993), no. 1, 67–73. MR 1190367 (93j:47005)

[BD71] F. F. Bonsall and J. Duncan, *Numerical ranges of operators on normed spaces and of elements of normed algebras*, London Mathematical Society Lecture Note Series, vol. 2, Cambridge University Press, London, 1971. MR 0288583 (44 #5779)

[BD73] ———, *Numerical ranges. II*, Cambridge University Press, New York, 1973, London Mathematical Society Lecture Notes Series, No. 10. MR 0442682 (56 #1063)

[Bol70] Béla Bollobás, *An extension to the theorem of Bishop and Phelps*, Bull. London Math. Soc. **2** (1970), 181–182. MR 0267380 (42 #2282)

[BS84] I. D. Berg and Brailey Sims, *Denseness of operators which attain their numerical radius*, J. Austral. Math. Soc. Ser. A **36** (1984), no. 1, 130–133. MR 720006 (84j:47004)

[Car85a] Carmen Silvia Cardassi, *Density of numerical radius attaining operators on some reflexive spaces*, Bull. Austral. Math. Soc. **31** (1985), no. 1, 1–3. MR 772627 (86e:47005)

[Car85b] ———, *Numerical radius attaining operators*, Banach spaces (Columbia, Mo., 1984), Lecture Notes in Math., vol. 1166, Springer, Berlin, 1985, pp. 11–14. MR 827753 (87c:47004)

[Car85c] ———, *Numerical radius-attaining operators on $C(K)$*, Proc. Amer. Math. Soc. **95** (1985), no. 4, 537–543. MR 810159 (87a:47066)

[KMP06] Vladimir Kadets, Miguel Martín, and Rafael Payá, *Recent progress and open questions on the numerical index of Banach spaces*, RACSAM. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. **100** (2006), no. 1-2, 155–182. MR 2267407 (2007h:46011)

[Lin63] Joram Lindenstrauss, *On operators which attain their norm*, Israel J. Math. **1** (1963), 139–148. MR 0160094 (28 #3308)

[Pay92] Rafael Payá, *A counterexample on numerical radius attaining operators*, Israel J. Math. **79** (1992), no. 1, 83–101. MR 1195254 (93j:47004)

[Sim72] B. Sims, *On numerical range and its applications to banach algegras*, Ph.D. thesis, University of Newcastle, Australia, 1972.

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